1 Method of moving frame

Let M^n be a manifold with geometric structure. The **method of moving frame** devised by E. Cartan consists of

(i) finding a natural frame $\theta = (\theta^1, \cdots, \theta^n)$ over M,

(ii) expressing $d\theta$ in terms of θ,

(iii) finding a complete system of invariants.

Example 1.1 (Frenet frame for a curve in $\mathbb{R}^3$).
Let $\beta(s)$ be a curve in $\mathbb{R}^3$ parameterized by arc length s. Let $T = \beta'(s)$ be a unit tangent vector. Then $T'(s) = k(s)N$ where N is a unit normal vector to β. Assume $k > 0$. Let $B = T \times N$ be the binormal vector. Then $B' = -\tau N$. We have the Frenet formula :

$$\begin{cases} T' &= kN, \\ N' &= -kT + \tau B, \\ B' &= -\tau N. \end{cases}$$

A pair (T, N, B) is a moving frame along the curve β. We expressed $(T, N, B)'$ in terms of (T, N, B) and obtained a complete system of invariants $\{k, \tau\}$. Suppose there are two curves $\alpha(s)$ and $\beta(s)$ in $\mathbb{R}^3$ which are parameterized by arc length s. If $k_\alpha = k_\beta$ and $\tau_\alpha = \tau_\beta$, then α and β are congruent.

Let G be a Lie group and $\mathfrak{g}$ the associated Lie algebra.

Definition 1.2. A **Maurer-Cartan form** is a $\mathfrak{g}$-valued 1-form ω which satisfies the Maurer-Cartan equation :

$$d\omega = -\frac{1}{2}[\omega, \omega],$$

that is, for any tangent vectors X and Y to G, $d\omega(X, Y) = -[\omega(X), \omega(Y)]$.

In the cases that G is a Lie subgroup of $Gl(n, \mathbb{R})$, let $g : G \hookrightarrow Gl(n, \mathbb{R})$ be the inclusion map. Then $\eta := g^{-1}dg$ is a Maurer-Cartan form. It has the following properties:

(i) η is a $\mathfrak{g}$-valued 1-form.
For $g_0 \in G$ and $v \in T_{g_0}G$,

$$\begin{aligned}
\eta(v) &= g_0^{-1}dg(v) \\
&= g_0^{-1}\frac{d}{dt}\Big|_{t=0} g(\alpha(t)) \ \text{ where } \alpha(0) = g_0,\ \alpha'(0) = v \\
&= \frac{d}{dt}\Big|_{t=0} g_0^{-1}g(\alpha(t)) \in T_eG = \mathfrak{g}.
\end{aligned}$$

(ii) η is left invariant.
For any $a \in G$,

$$\begin{aligned}
L_a^*\eta &= (ag)^{-1}d(ag) \\
&= g^{-1}a^{-1}adg \\
&= g^{-1}dg \\
&= \eta.
\end{aligned}$$

(iii) $d\eta = -\frac{1}{2}[\eta, \eta]$.
Since $g^{-1}g = I$, $(dg^{-1})g + g^{-1}dg = 0$. Thus $d(g^{-1}) = -g^{-1}(dg)g^{-1}$.
Now,

$$\begin{aligned}
d\eta &= d(g^{-1}) \wedge dg + g^{-1}ddg \\
&= -g^{-1}dgg^{-1}dg \\
&= -\eta \wedge \eta.
\end{aligned}$$

In $\mathbb{R}^n$, a frame is an ordered set of vectors

$$F = (x, e_1, \ldots, e_n),$$

where $x \in \mathbb{R}^n$ and e_i's are orthonormal tangent vectors at x. Such frames form the group $E(n)$ of Euclidean motions. $E(n)$ is the set of all the matrices of the form

$$\begin{bmatrix} 1 & 0 & \cdots & 0 \\ x_1 & & & \\ \vdots & e_1 & \cdots & e_n \\ x_n & & & \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ x & A \end{bmatrix} \quad \text{where } {}^tAA = I.$$

$E(n)$ is a group since

$$\begin{bmatrix} 1 & 0 \\ x & A \end{bmatrix} \begin{bmatrix} 1 & 0 \\ y & B \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ x + Ay & AB \end{bmatrix},$$

$$\begin{bmatrix} 1 & 0 \\ x & A \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 \\ -{}^tAx & {}^tA \end{bmatrix}.$$

Now compute the Maurer-Cartan form of $E(n)$.
Let $g : E(n) \hookrightarrow Gl(n+1, \mathbb{R})$ be the inclusion. Then we have

$$\begin{aligned} \eta &= g^{-1}dg \\ &= \begin{bmatrix} 1 & 0 \\ -{}^tAx & {}^tA \end{bmatrix} \begin{bmatrix} 0 & 0 \\ dx & dA \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 \\ {}^tAdx & {}^tAdA \end{bmatrix}. \end{aligned}$$

From the bundle point of view, $\pi : E(n) \longrightarrow \mathbb{R}^n$ given by $(x, e_1, \cdots, e_n) \mapsto x$ gives a principal fibration :

$$E(n) \xrightarrow{\pi} E(n)/O(n) \approx \mathbb{R}^n$$

with structure group $O(n)$. Let $\sigma(x) = (x, e_1, \cdots, e_n)$ be an orthonormal frame field. Pull back η by σ. Then we obtain

$$\sigma^*\eta = \begin{bmatrix} 0 & 0 \\ \theta & \omega \end{bmatrix} = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ \theta_1 & & & \\ \vdots & & \omega^i_j & \\ \theta_n & & & \end{bmatrix}.$$

In order to find θ and ω, we let $A = \begin{bmatrix} e_1 & \cdots & e_n \end{bmatrix}$. Then ${}^tAdx = \theta$ and ${}^tAdA = \omega$. Observe that the Maurer-Cartan equation $d\eta + \eta \wedge \eta = 0$ implies

$$\begin{aligned}
0 &= \sigma^*(d\eta + \eta \wedge \eta) \\
&= d(\sigma^*\eta) + \sigma^*\eta \wedge \sigma^*\eta \\
&= \begin{bmatrix} 0 & 0 \\ d\theta & d\omega \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ \theta & \omega \end{bmatrix} \wedge \begin{bmatrix} 0 & 0 \\ \theta & \omega \end{bmatrix} \\
&= \begin{bmatrix} 0 & 0 \\ d\theta + \omega \wedge \theta & d\omega + \omega \wedge \omega \end{bmatrix}.
\end{aligned}$$

Hence $d\theta + \omega \wedge \theta = 0$ and $d\omega + \omega \wedge \omega = 0$.

In general, let G be a Lie group of $Gl(n, \mathbb{R})$ and H a closed subgroup of G. G may be regarded as the set of frames of G/H. Then the Maurer-Cartan forms appear in the structure equations of a moving frame. The Maurer-Cartan equations give a complete set of relations for the structure equations of a moving frame. The question of describing the position of a submanifold $M \subset G/H$ may be thought of attaching to M a natural frame or equivalently, a cross section of $G \to G/H$ over M. The Maurer-Cartan form for G when restricted to the natural frame becomes a complete set of invariants for M in G/H.

2 Local geometry of a submanifold $M^m \subset \mathbb{R}^N$

Choose a natural frame(adapted frame) of M, that is, an orthonormal frame $e_1, \ldots, e_m, e_{m+1}, \ldots, e_N$ where $e_1, \ldots, e_m$ are tangent to M. Let $\sigma : M \to E(N)$ be the map $x \mapsto (x, e_1, \ldots, e_N)$.

Definition 2.1. Pull back by σ of the Maurer-Cartan form η of $E(N)$ is called the **complete system of (local) invariants** of M with respect to the group of euclidean motions.

Let M, M' be submanifolds of $\mathbb{R}^N$ of dimension m. Let σ, σ' be adapted frames and $(\theta, \omega), (\theta', \omega')$ the complete systems of local invariants of M and M', respectively. Then M and M' are congruent if and only if there exists $\varphi : M \to M'$ such that $\varphi^*(\theta', \omega') = (\theta, \omega)$.

The previous example of Frenet frame is a special case :

$$k = \omega_2^1(T), \quad \tau = \omega_3^2(T).$$

Now we consider the existence and uniqueness of maps into Lie groups. Let G be a Lie group with the Lie algebra $\mathfrak{g} = T_e G$ and let η be a $\mathfrak{g}$-valued 1-form on G such that

(i) $\eta_e : \mathfrak{g} \longrightarrow \mathfrak{g}$ is the identity map,

(ii) $L_a^* \eta = \eta \quad$ for all $a \in G$.

Such an η exists and is unique. If $G \hookrightarrow Gl(n, \mathbb{R})$, then $\eta = g^{-1}dg$ is such a Maurer-Cartan form.

The following theorem is due to E. Cartan.

Theorem 2.2. *Let N be a connected and simply connected manifold and let γ be a smooth $\mathfrak{g}$-valued 1-form on N such that $d\gamma = -\frac{1}{2}[\gamma, \gamma]$. Then, there is a smooth map $g : N \to G$ which is unique up to composition with a constant left multiplication so that $g^*\eta = \gamma$.*

Proof. We assume that G and $\mathfrak{g}$ are matrix groups. Let $M = N \times G$ and consider a 1-form $\theta = \eta - \gamma$. Then we have

$$\begin{aligned} d\theta &= d\eta - d\gamma \\ &= -\eta \wedge \eta + \gamma \wedge \gamma \\ &= -(\theta + \gamma) \wedge (\theta + \gamma) + \gamma \wedge \gamma \\ &= -\theta \wedge (\theta + \gamma) - \gamma \wedge \theta. \end{aligned}$$

We write $\theta = \theta^1 x_1 + \cdots + \theta^s x_s$ where $\{x_1, \ldots, x_s\}$ is a basis of $\mathfrak{g}$ and $\theta^1, \ldots, \theta^s$ are 1-forms on M. Then the algebraic ideal $I = < \theta^1, \ldots, \theta^s >$

satisfies $dI \subset I$. Moreover, $\theta^1, \ldots, \theta^s$ are linearly independent because they restrict to each fibre $\{n\} \times G$ to be linearly independent. By Frobenius theorem, M is foliated by maximal connected integral manifolds of I, each of which projects onto N to be a covering map. Observe that the foliation is invariant under $Id \times L_a : N \times G \to N \times G$ since η is left invariant. Since N is connected and simply connected, each integral leaf projects diffeomorphically onto N and hence the graph of a map $g : N \to G$. $\square$

References

[1] R. Bryant, S. S. Chern, R. Gardner, H. Goldschmidt and P. Griffiths, *Exterior differential systems* (1991), Springer-Verlag, New York, Berlin, Heidelberg.

[2] R. Bryant, P. Griffiths and D. Yang, *Characteristics and existence of isometric embeddings*, Duke Math. J. **50** (1983), 893-994.